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Predictor-corrector scheme for solving second order ordinary differential equations

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ABSTRACT

In this research, the direct method of Adam Moulton two-step method was proposed for solving initial value problem (IVPs) of second order ordinary differential equations (ODEs) directly. The current approach for solving second order ODEs is to reduce to first order ODEs. However, the direct method in this research will solved the second order ODEs directly. The Lagrange interpolation polynomial was applied in the derivation of the proposed method. The implementation will be in predictor-corrector scheme. Numerical results shown that the method gave comparable accuracy and faster execution time compared to the existing method. The proposed direct method of Adams Moulton type is suitable for solving second order ODEs.

Keywords: Direct method, two-step method, second order ODEs, predictor-corrector method, Van der Pol's problem

INTRODUCTION

Ordinary differential equations (ODEs) are very important to solve sciences and engineering problems. For example, ODEs are used in Newton's second law and law of cooling. ODEs are also used in Hooke's law for modeling the motion of a spring and in modeling population growth and exponential decay. In engineering, ODEs are used in various field such as mechanical vibration, dynamical systems theory and in theory of electrical circuit.

In this research, the general form of initial value problem (IVPs) for the second order ODEs will be considered as follows:

$$y'' = f(t, y, y'),\ y(a) = y_0,\ \ y^{'}(a) = y_0',\ \ a \le t \le b \qquad (1)$$

where y_0 and y_0' are initial values and f is a continuous function. This IVPs will be solved by the proposed direct method without reduced to first order ODEs that will reduce computation cost. There are several methods that can be used to solve the second order ODEs numerically but need to reduce to first order ODEs, such as Runge-Kutta (RK) method and Adams Bashforth-Moulton (ABM) method. The ABM method also known as predictor-corrector method.

Van Der Houmen and Someijer (1987) has proposed predictor-corrector method for second order ODEs. Their proposed method has order four and five and phase error of orders up to ten. Khiyal and Thomas (1997) proposed a variable order and variable step algorithm to solve second order IVP. Numerical method of solving second order IVPs directly with step length k = 4 based on collocation of the differential system and interpolation of the approximate solution has been introduced by Adesanya et al. (2008).

Badmus and Yahaya (2009) developed a uniform order 6 of five-step block methods for direct solution of general second order ODEs. Anake (2011) proposed a modified developed one-step implicit block methods where the proposed method gave very low error terms. Modified block method base on the collocation and interpolation of the power series approximation has been proposed by Awoyemi et al. (2011).

Two-point four step direct implicit block method has been developed by Majid et al. (2009) for solving the second order ODEs directly using variable step-size. This method will estimate the solutions of IVPs at two points simultaneously by using four backward steps. Majid et al. (2012) proposed direct two-point block one-step method to solve second order ODEs directly. The result is faster compared to the existing method.

Omar and Alkasassbeh (2016) studied on generalize implicit one-step third derivative block method for solving second order ODEs directly using collocation and interpolation approach. The approximate solution of the power series is interpolated at the first and off-step points.

The aim of this research is to propose the direct method of Adams Bashforth-Moulton of two-step in the predictor-corrector mode for solving the second order ODEs directly.

FORMULATION

Derivation of the method

The corrector formula of the direct method will be derived in this section. The point, y_{m+1} at t_{m+1} can be obtained by integrating equation (1) over the interval $[\,t_m, t_{m+1}]$. Integrate once we have:

$$\int_{t_m}^{t_{m+1}} y''(t)dt = \int_{t_m}^{t_{m+1}} f(t,y,y')dt$$

Therefore,

$$y'(t_{m+1}) - y'(t_m) = \int_{t_m}^{t_{m+1}} f(t,y,y')dt \qquad (2)$$

Integrate twice, we have

$$\int_{t_m}^{t_{m+1}} \int_{t_m}^{t} y''(t)dtdt = \int_{t_m}^{t_{m+1}} \int_{t_m}^{t} f(t,y,y')dtdt$$

Therefore,

$$y(t_{m+1}) - y(t_m) - hy'(t_m) = \int_{t_m}^{t_{m+1}} \int_{t_m}^{t} f(t,y,y')dtdt \qquad (3)$$

By replacing $f(t,y,y')$ in equation (2) and (3) with the polynomial interpolation at the points $\{(t_{m-1}, f_{m-1}), (t_m, f_m), (t_{m+1}, f_{m+1})\}$, we have

$$y'(t_{m+1}) - y'(t_m) = \int_{t_m}^{t_{m+1}} [\frac{(t-t_m)(t-t_{m+1})}{(t_{m-1}-t_m)(t_{m-1}-t_{m+1})} \quad f_{m-1} + \frac{(t-t_{m-1})(t-t_{m+1})}{(t_m-t_{m-1})(t_m-t_{m+1})} \quad f_m + \frac{(t-t_{m-1})(t-t_m)}{(t_{m+1}-t_{m-1})(t_{m+1}-t_m)} \quad f_{m+1}]dt \qquad (4)$$

$$y(t_{m+1}) - y(t_m) - hy'(t_m) = \int_{t_m}^{t_{m+1}} (t_{m+1}-t)[\frac{(t-t_m)(t-t_{m+1})}{(t_{m-1}-t_m)(t_{m-1}-t_{m+1})} \quad f_{m-1} + \frac{(t-t_{m-1})(t-t_{m+1})}{(t_m-t_{m-1})(t_m-t_{m+1})} \quad f_n + \frac{(t-t_{m-1})(t-t_m)}{(t_{m+1}-t_{m-1})(t_{m+1}-t_m)} \quad f_{m+1} \ ]dt \qquad (5)$$

By taking $s = \frac{t-t_{m+1}}{h}$ and replacing $dt = hds$, the corrector formulae can be obtained by integrating (4) and (5).

Corrector formulae:

$$y'_{m+1} = y'_m + \frac{h}{12}(5f_{m+1} + 8f_m - f_{m-1})$$
$$y_{m+1} = y_m + hy'_m + \frac{h^2}{24}(3f_{m+1} + 10f_m - f_{m-1}). \qquad (6)$$

The predictor formula (7) can be found in Majid et al. (2011), where the author has developed the method and extend it to solve the linear boundary value problem.

Predictor formulae:

$$y'_{m+1} = y'_m + \frac{h}{12}(23f_m - 16f_{m-1} + 5f_{m-2})$$
$$y_{m+1} = y_m + hy'_m + \frac{h^2}{24}(19f_m - 10f_{m-1} + 3f_{m-2}). \qquad (7)$$

Order of the method

The order of the method will be discussed in this section, and the formula (6) is display in a matrix form as follows:

From Eq. (6), we have

$$0 = -y'_{m+1} + y'_m + \frac{h}{12}(5f_{m+1} + 8f_m - f_{m-1})$$
$$y_{m+1} - y_m = hy'_m + \frac{h^2}{24}(3f_{m+1} + 10f_m - f_{m-1}).$$

Then we have

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} y_{m-1} \\ y_m \\ y_{m+1} \end{pmatrix} = h \begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} y'_{m-1} \\ y'_m \\ y'_{m+1} \end{pmatrix} + h^2 \begin{pmatrix} -\frac{1}{12} & \frac{8}{12} & \frac{5}{12} \\ -\frac{1}{24} & \frac{10}{24} & \frac{3}{24} \end{pmatrix} \begin{pmatrix} f_{m-1} \\ f_m \\ f_{m+1} \end{pmatrix}.$$

Next, we substitute in linear difference operator, and the derivatives are extended by Taylor series. The coefficient matrix C is obtained as follows:

$$C_0 = \alpha_0 + \alpha_1 + \alpha_2 = \begin{pmatrix}0\\0\end{pmatrix} + \begin{pmatrix}0\\-1\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$C_1 = \alpha_1 + 2\alpha_2 - (\beta_0 + \beta_1 + \beta_2) = \begin{pmatrix}0\\-1\end{pmatrix} + 2\begin{pmatrix}0\\1\end{pmatrix} - \begin{pmatrix}1\\1\end{pmatrix} - \begin{pmatrix}-1\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$C_2 = \frac{\alpha_1}{2} + 2\alpha_2 - (\beta_1 + \beta_2) - (\gamma_0 + \gamma_1 + \gamma_2) = \frac{1}{2}\begin{pmatrix}0\\-1\end{pmatrix} + 2\begin{pmatrix}0\\1\end{pmatrix} - \begin{pmatrix}-1\\1\end{pmatrix} - \begin{pmatrix}1\\1/2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$C_3 = \frac{\alpha_1}{6} + \frac{8\alpha_2}{6} - \left(\frac{\beta_1}{2} + 2\beta_2\right) - (\gamma_1 + 2\gamma_2) = \frac{1}{6}\begin{pmatrix}0\\-1\end{pmatrix} + \frac{8}{6}\begin{pmatrix}0\\1\end{pmatrix} - \begin{pmatrix}-3/2\\1/2\end{pmatrix} - \begin{pmatrix}18/12\\16/24\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$C_4 = \frac{\alpha_1}{24} + \frac{16\alpha_2}{24} - \left(\frac{\beta_1}{6} + \frac{8\beta_2}{6}\right) - \left(\frac{\gamma_1}{2} + 2\gamma_2\right) = \begin{pmatrix}0\\15/24\end{pmatrix} - \begin{pmatrix}-7/6\\1/6\end{pmatrix} - \begin{pmatrix}14/12\\11/24\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$C_5 = \frac{\alpha_1}{120} + \frac{32\alpha_2}{120} - \left(\frac{\beta_1}{24} + \frac{16\beta_2}{24}\right) - \left(\frac{\gamma_1}{6} + \frac{8\gamma_2}{6}\right) = \begin{pmatrix}0\\ \frac{31}{120}\end{pmatrix} - \begin{pmatrix}-\frac{15}{24}\\ \frac{1}{24}\end{pmatrix} - \begin{pmatrix}\frac{48}{72}\\ \frac{34}{144}\end{pmatrix} = \begin{pmatrix}-1/24\\-7/360\end{pmatrix}$$

Lambert (1973) stated that the method is of order n if $C_0 = C_1 = \cdots = C_m = C_{m+1} = 0$ and $C_{m+2} \neq 0.$ Since $C_5 \neq 0$, hence, the proposed method is order three and known as Direct Adam Moulton two-step method of order three (DAM2SM3).

Consistency of the method

Let Z_j , Z'_j and Z''_j below represent the matrices of theoretical solutions for ODE (1),

$$Z_j = \begin{pmatrix}y(t_{j-1})\\ y(t_j)\\ y(t_{j+1})\end{pmatrix}, Z'_j = \begin{pmatrix}y'(t_{j-1})\\ y'(t_j)\\ y'(t_{j+1})\end{pmatrix} \text{ and } Z''_j = \begin{pmatrix}f(t_{j-1}, y(t_{j-1}), y'(t_{j-1}))\\ f(t_j, y(t_j), y'(t_j))\\ f(t_{j+1}, y(t_{j+1}), y'(t_{j+1}))\end{pmatrix}$$

Fatunla (1991) stated that the local truncation error for linear multistep method is introduced as

$$E_j = \alpha Z_j - h\beta Z'_j - h^2\gamma Z''_j$$
$$\|E_j\| = \alpha Z_j - h\beta Z'_j - h^2\gamma Z''_j$$

where $\|.\|$is the maximum norm. The maximum norm of LTE for the DAM2SM3 is

$$\|E_j\| = h^5\begin{pmatrix}-\frac{1}{24}\\ -\frac{7}{360}\end{pmatrix}.$$

As the step-size h moving toward zero, then the DAM2SM3 method is consistent, thus $\|E_j\|$ is also tends to zero.

Zero stability of the method

Zero stability is needed to declare the stability of the method at the chosen step-size. The first characteristic polynomial $\varphi(R)$ is introduced:

$$\varphi(R) = \det(A_0 R - A_1) = 0$$

have the roots R_j that satisfy $|R_j| \leq 1$ and the roots have multiplicity not exceeding 2 for $|R_j| = 1$, then the DAM2SM3 is considered as zero stable.

$$\varphi(N) = \det\begin{pmatrix} R-1 & 0 \\ 0 & R-1 \end{pmatrix} = 0,$$

$$(R-1)^2 = 0, \qquad R = 1,1$$

where $A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then DAM2SM3 is zero stable.

Convergence of the method

The Dahlquist convergence theorem states that if linear multistep method is zero stable and consistent, then it is a convergent method. The proposed method DAM2SM3 is converges to the accurate solution since it has attained consistency and zero stability.

Stability Region

The stability region will be obtained by substitute the test equation

$$y'' = f = \lambda y' + \beta y$$

into the proposed method (6). We may get as follows

$$y'_{m+1} = y'_m + \frac{5}{12}h\lambda y'_{m+1} + \frac{5}{12}h\beta y_{m+1} + \frac{8}{12}h\lambda y'_m + \frac{8}{12}h\beta y_m - \frac{1}{12}h\lambda y'_{m-1} - \frac{1}{12}h\beta y_{m-1}.$$

$$y_{m+1} = y_m + hy'_m + \frac{3}{24}h^2\lambda y'_{m+1} + \frac{3}{24}h^2\beta y_{m+1} + \frac{10}{24}h^2\lambda y'_m + \frac{10}{24}h^2\beta y_m - \frac{1}{24}h^2\lambda y'_{m-1} - \frac{1}{24}h^2\beta y_{m-1}.$$

Then we change equations above in matrix form

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y'_{m+1} \\ y_{m+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y'_m \\ y_m \end{pmatrix} + h\begin{pmatrix} \frac{5}{12}\lambda & \frac{5}{12}\beta \\ 0 & 0 \end{pmatrix}\begin{pmatrix} y'_{m+1} \\ y_{m+1} \end{pmatrix} + h\begin{pmatrix} \frac{2}{3}\lambda & \frac{2}{3}\beta \\ 1 & 0 \end{pmatrix}\begin{pmatrix} y'_m \\ y_m \end{pmatrix}$$

$$+h\begin{pmatrix} -\frac{1}{12}\lambda & -\frac{1}{12}\beta \\ 0 & 0 \end{pmatrix}\begin{pmatrix} y'_{m-1} \\ y_{m-1} \end{pmatrix} + h^2\begin{pmatrix} 0 & 0 \\ \frac{1}{8}\lambda & \frac{1}{8}\beta \end{pmatrix}\begin{pmatrix} y'_{m+1} \\ y_{m+1} \end{pmatrix}$$

$$+h^2\begin{pmatrix} 0 & 0 \\ \dfrac{5}{12}\lambda & \dfrac{5}{12}\beta \end{pmatrix}\begin{pmatrix} y'_{m} \\ y_{m} \end{pmatrix} + h^2\begin{pmatrix} 0 & 0 \\ -\dfrac{1}{24}\lambda & -\dfrac{1}{24}\beta \end{pmatrix}\begin{pmatrix} y'_{m-1} \\ y_{m-1} \end{pmatrix}$$

From above matrix, we have

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$B_0 = \begin{pmatrix} \dfrac{5}{12}\lambda & \dfrac{5}{12}\beta \\ 0 & 0 \end{pmatrix}, B_1 = \begin{pmatrix} \dfrac{2}{3}\lambda & \dfrac{2}{3}\beta \\ 1 & 0 \end{pmatrix}, B_2 = \begin{pmatrix} -\dfrac{1}{12}\lambda & -\dfrac{1}{12}\beta \\ 0 & 0 \end{pmatrix}$$

$$C_0 = \begin{pmatrix} 0 & 0 \\ \frac{1}{8}\lambda & \frac{1}{8}\beta \end{pmatrix}, C_1 = \begin{pmatrix} 0 & 0 \\ \frac{5}{12}\lambda & \frac{5}{12}\beta \end{pmatrix}, C_2 = \begin{pmatrix} 0 & 0 \\ -\frac{1}{24}\lambda & -\frac{1}{24}\beta \end{pmatrix}.$$

General equation of stability is as follows,

$$\sum_{k=0}^{r} A_k Y_{m-k} + h\sum_{k=0}^{r+1} B_k Y_{m-k} + h^2\sum_{k=0}^{r+1} C_k Y_{m-k} = 0$$

By substituting the value r = 1, we have

$$Y_m(A_0 + hB_0 + h^2C_0) - Y_{m-1}(A_1 + hB_1 + h^2C_1) - Y_{m-2}(hB_2 + h^2C_2) = 0$$

Solving the determinant of

$$w^2(A_0 - hB_0 - h^2C_0) - w(A_1 + hB_1 + h^2C_1) - (hB_2 + h^2C_2) = 0$$

By substitute $Y = h^2\beta$ and $X = h\lambda$, the stability polynomial is obtained

$$\frac{24w^4 - 10Xw^4 - 3Yw^4 - 6Xw^3 - 17Yw^3 - 48w^3 + 18Xw^2 - 5Yw^2 + 24w^2 - 2Xw + Yw}{24} = 0.$$

The boundary of the stability region in $X - Y$ plane is determined by substituting the values of $w = 0, -1$ and $e^{i\theta}$_ where $0 \leq \theta \leq 2\pi$ into stability polynomial. The stability region of the direct method will be shown in Figure 1 and the bounded shaded is the stable region.

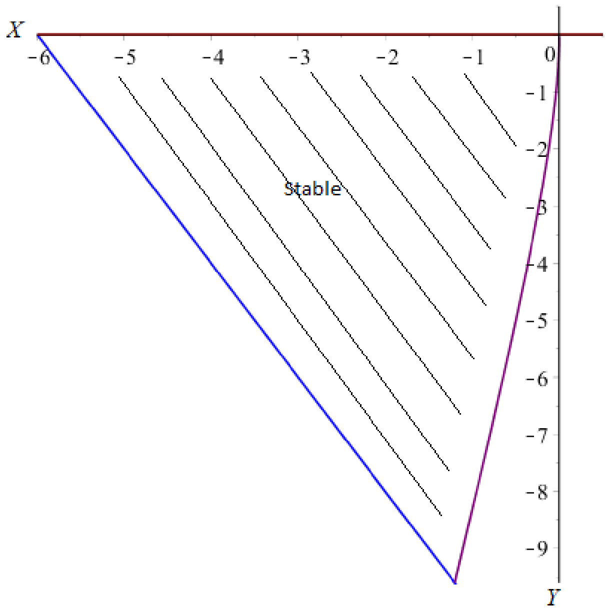

Figure 1: Stability region for DAM2SM3

IMPLEMENTATION

The implemented program began by using direct Euler method only at the beginning of the interval to calculate the three starting initial points. Then the proposed method DAM2SM3 will used the initial points and continue the calculation until the end of the interval.

Algorithm of DAM2SM3

Step 1: Set starting value *a*, ending value *T*, step-size *h* and given initial value.

Step 2: Use direct Euler method to find f_0, f_1 and f_2.

Step 3: While $t_n < T$, do Step 4 and Step 5.

Step 4: Calculate y'_{m+1} and y_{m+1} using the predictor formulae in (7).

Step 5: Calculate y'_{m+1} and y_{m+1} using the corrector formulae in (6).

Step 6: Complete.

NUMERICAL RESULTS

Five numerical examples of second order ODEs problem were tested to study the capability of the DAM2SM3 method. All the programs were written in C language.

The following abbreviations are used in the tables which summarize the numerical results.

h	Step size
FCN	Total function calls
MAXE	Maximum error

TIME	Timing in second
DAM2SM3	Direct Adam Moulton two-step method of order three
BI	Backward difference with the reduction to first order system (Rasedee (2014))
ABM3	Adam Moulton two-step method of order three
RM1S (3)	Third order one-step rational method (Fairuz and Majid (2021))

Problem 1. Consider

$$y'' - x(y')^2 = 0, y(0) = 1, y'(0) = \frac{1}{2}, [0,1].$$

Exact solution:

$$y(x) = 1 - \frac{1}{2}\ln\left(\frac{2+x}{2-x}\right).$$

Table 1: Results of DAM2SM3 and ABM3 for Problem 1

h	METHOD	MAXE	FCN	TIME
0.1	DAM2SM3	2.7246e-003	19	0.059
	ABM3	2.2118e-003	20	0.063
0.01	DAM2SM3	3.0071e-005	199	0.060
	ABM3	2.2213e-005	200	0.113
0.001	DAM2SM3	3.0366e-007	1999	0.071
	ABM3	2.2221e-007	2000	0.610

Problem 2. Consider

$$y' + \lambda^2 y = 0, \text{ we take } \lambda = 2, y(0) = 1, y'(0) = 2, [0,1].$$

Exact solution:

$$y(x) = \cos 2x + \sin 2x.$$

Table 2: Results of DAM2SM3 and ABM3 for Problem 2

h	METHOD	MAXE	FCN	TIME
0.1	DAM2SM3	3.2919e-002	19	0.051
	ABM3	4.5170e-002	38	0.041
0.01	DAM2SM3	3.9336e-004	199	0.054
	ABM3	5.2230e-004	398	0.110
0.001	DAM2SM3	3.9934e-006	1999	0.064
	ABM3	5.2938e-006	3998	0.683

Problem 3. Consider

$$y'' = 2y' - y, \qquad y(0) = 0, y'(0) = 1, [0,1].$$

Exact solution:

$$y(x) = xe^x.$$

Table 3: Results of DAM2SM3 and ABM3 for Problem 3

h	METHOD	MAXE	FCN	TIME
0.1	DAM2SM3	6.6979e-002	19	0.044
	ABM3	5.2267e-002	38	0.042
0.01	DAM2SM3	8.0049e-004	199	0.054
	ABM3	5.4182e-004	398	0.121
0.001	DAM2SM3	8.1398e-006	1999	0.062
	ABM3	5.4347e-006	3998	0.655

Problem 4. Consider

$$y'' = -y + 2e^{-x} + 1, y(0) = 3, y'(0) = 0, [0,1].$$

Exact solution:

$$y(x) = \cos 2x + \sin 2x.$$

Table 4: Results of DAM2SM3 and ABM3 for Problem 4

h	DAM2SM3	MAXE	FCN	TIME
0.1	ABM3	1.4231e-002	19	0.066
	DAM2SM3	9.6666e-003	38	0.059
0.01	ABM3	1.6563e-004	199	0.057
	DAM2SM3	9.9666e-005	398	0.136
0.001	ABM3	1.6802e-006	1999	0.084
	DAM2SM3	9.9966e-007	3998	0.669

Problem 1- 4 have been solved using three different step sizes h = 0.1, 0.01 and 0.001. Tables 1 – 4 shown the comparison results between the accuracy result using DAM2SM3 and ABM3. Based on the results, the DAM2SM3 is comparable compared to ABM3 in term of maximum error. In term of total function calls, DAM2SM3 is lesser compared to ABM3. The execution times for DAM2SM3 when solving the tested problems is faster compared to ABM3. The numerical results in the Tables show that the DAM2SM3 is suitable for solving second order ODEs directly compared to ABM3.

Problem 5. Van der Pol oscillator:

$$y'' - \alpha(1 - y^2)y' + y = 0, y(0) = 2, y'(0) = 0, [0,1], \alpha = 5.$$

Direct method DAM2SM3 was tested on Van der Pol's problem by taking $\alpha = 5$. This equation applied widely in physics (oscillatory phenomenon), electronic and chemical reaction study. The approximate solutions at the end point $y(1)$ for Van der Pol's problem are given in Table 5. The solutions of Problem 5 when $h = 0.01$ are also plotted and shown in Figure 2.

Table 5: Result of DAM2SM3 for Problem 5

h	Method	$y(1)$

0.1	DAM2SM3 RM1S(3)	1.87791253e+00 1.86765499e+00
0.01	DAM2SM3 RM1S(3)	1.86923623e+00 1.86945094e+00
0.001	DAM2SM3 RM1S(3)	1.86943665e+00 1.86943904e+00
0.0001	DAM2SM3 RM1S(3)	1.86943883e+00 1.86943885e+00
Tolerance		
10^{-2}	BI	1.878019825e+00
10^{-4}	BI	1.869403059e+00
10^{-6}	BI	1.869436891e+00
10^{-8}	BI	1.869440272e+00

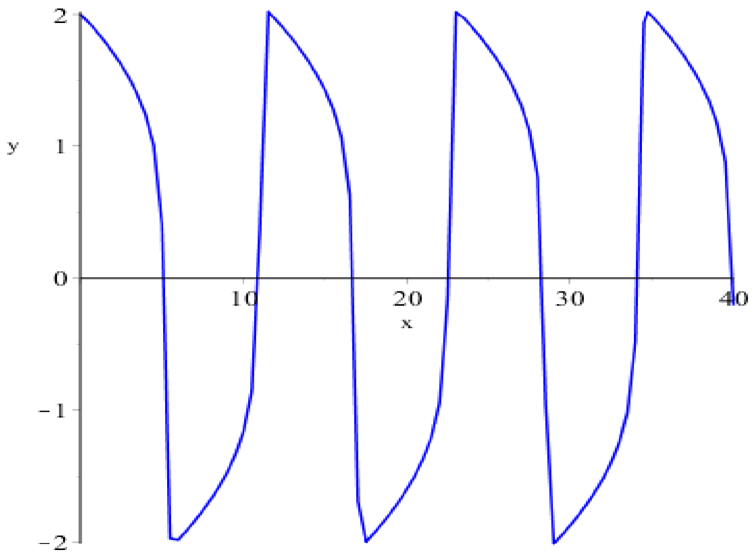

Figure 2: Plot of y (approximation solutions) versus values of x when $h = 0.01$ using DAM2SM3

In Table 5, the approximate solutions of DAM2SM3 for solving the Van der Pol's problem were compared to RM1S(3) and BI methods at different values of step-size. As the step-size decreased, the DAM2SM3 method is able to produce approximate solutions of $y(1)$ comparable to the values given by RM1S(3) and BI methods.

CONCLUSION

Direct Adam Moulton two-step method of order three (DAM2SM3) based on predictor-corrector scheme has been developed for solving second order derivative of ODEs directly. The accuracy of DAM2SM3 for solving the tested problems improved as the step-sizes decreased. The proposed direct method has shown less expensive in terms of total function calls and timing because the method has avoided the strategy of reducing to first order ODEs approach because it will enlarge the systems of first order equations. The DAM2SM3 also has been tested on Van der Pol's problem and the results are in good agreement to the existing methods in terms of accuracy.

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